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A Simplicial Algorithm for Testing the Integral Property of a Polytope

Zaifu Yang *

Abstract Given an n-dimensional simplex

$$P = \{ x \in R^n \mid a_i^\top x \leq b_i, i = 1, ..., n + 1 \}$$

where R^n is the n-dimensional Euclidean space, the question is to determine whether P contains an integral point or not. We propose a simplicial algorithm to answer the question based on a specific integer labeling rule and the K_1-triangulation of R^n. Starting from an arbitrary integral point of R^n, the algorithm terminates within a finite number of steps with either an integral point in P or proving there is no integral point in P. The algorithm also applies to general polytopes. One prominent feature of the algorithm is that the structure of the algorithm is very simple and it can be easily implemented on a computer. Moreover, the algorithm is computationally very simple, flexible and stable.

Keywords: Polytope, integral point, simplicial method, integer linear programming.

1 Introduction

Given a polytope P, for example, the convex hull of $n + 1$ affinely independent vectors of R^n, the question is to determine whether P contains an integral point or

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not. We develop a simplicial algorithm to solve the problem. The algorithm is based on a specific integer labeling rule and the K_1-triangulation of R^n. The main feature of the algorithm can be described as follows: The algorithm subdivides R^n into n-dimensional simplices such that all integral points of R^n are the vertices of the triangulation, and then assigns an integer to each integral point of R^n according to the labeling rule. Starting from an arbitrary integral point, the algorithm generates a sequence of adjacent simplices of varying dimension and terminates with either the YES or (exclusively) NO answer within a finite number of steps. In YES case, the algorithm finds an integral point in P. The NO answer shows that there is no integral point in P.

Our work was motivated by the works of Scarf [11] and of Dang and van Maaren [1]. However, Scarf's algorithm is based on primitive sets. Although Dang and van Maaren's algorithm is also based on simplices, it does not guarantee to provide the YES or NO answer. Our algorithm and labeling rule are very much different from Dang and van Maaren's. We would also like to point out that our algorithm could date back to the work of van der Laan and Talman [5], although their algorithm was introduced to compute a fixed point of a continuous function.

The remainder of the paper is summarized next. In Section 2 the labeling rule and basic theorems are introduced. Section 3 gives a full description of the algorithm. In Section 4 we shall demonstrate how to transform the problems into the standard form. In Section 5 we apply the algorithm to general n-dimensional polytopes. In Section 6 we deal with lower dimensional polytopes. Concluding remarks are found in Section 7.

2 Integer labeling Rule

The problem is to test the integral property of an n-dimensional simplex P given by

$$P = \{ x \in R^n \mid Ax \leq b \},$$

where $a_i^\top = (a_{i1}, ..., a_{in})$ is the i-th row of the $n+1$ by n matix A for $i = 1, \cdots, n+1$, and $b = (b_1, \cdots, b_{n+1})^\top$ is a vector of R^{n+1}. Without loss of generality we assume that $a_1, \cdots, a_{n+1}$ are integral vectors of R^n, and $b = (b_1, \cdots, b_{n+1})^\top$ is an integral vector of R^{n+1}. Notice that since the polytope P is full dimensional, the origin of R^n is contained in the interior of the convex hull of the vectors $a_1, \cdots, a_{n+1}$. As usual, Z^n denotes the set of all integral points in R^n. Let N denote the set $\{ 1, ..., n+1 \}$ and N_{-i} the set N without the index i, for $i \in N$. Compared with the labeling rules of Scarf [11] and of Dang and van Maaren [1], We introduce the following labeling rule.

Labeling Rule: We assign $x \in Z^n$ with the label $l(x) = i$ if i is the smallest index for which

$$a_i^\top x - b_i = \max\{ a_j^\top x - b_j \mid a_j^\top x - b_j > 0, j \in N \}.$$

If $a_i^\top x \leq b_i$ for all $i = 1,...,n+1$, then $l(x) = 0$.

Notice that if $l(x) = 0$, then P contains at least one integral point. Let $\mathcal{T}$ be the K_1-triangulation of R^n to be described in the next section. This simplicial subdivision of R^n is such that the collection of the vertices of simplices in $\mathcal{T}$ is the set of all integral points of R^n. We denote a simplex with vertices $x^1,...,x^{n+1}$ by $\sigma(x^1, ..., x^{n+1})$. Given an n-dimensional simplex $\sigma(x^1, ..., x^{n+1})$ in $\mathcal{T}$, let

$$L(\sigma) = \{ l(x^1), ..., l(x^{n+1}) \}.$$

An n-simplex σ is called a completely labeled simplex if $|L(\sigma)| = n+1$. Specifically, an n-simplex σ is called a completely labeled simplex of type I if $L(\sigma) = \{ 0 \} \bigcup N_{-i}$

for an index $i \in N$. Whereas an n-simplex σ is called a completely labeled simplex of type II if $L(\sigma) = N$. Observe that a completely labeled simplex of type I has a vertex being an integral point in P.

Now we state our basic results.

Theorem 2.1 *The Labeling Rule results in at least one completely labeled simplex.*

Proof: It immediately follows from induction. We omit the details. □

Furthermore, one can derive the following sharper and more important results.

Theorem 2.2 *If P does not contain any integral point, then the Labeling Rule results in a unique completely labeled simplex.*

Clearly, the unique completely labeled simplex must be of type II. A proof of the above theorem is deferred to Section 4. This theorem can be seen as a generalization of the following lemma (see van der Laan [4] and Talman [14]).

Lemma 2.3 *Choose an arbitrary point $c \in R^n$. We assign $x \in Z^n$ with the label $l(x) = i$ if i is the smallest index for which*

$$x_i - c_i = \max\{\, x_j - c_j \,|\, x_j - c_j > 0,\ j \in N \,\}.$$

If $x_i \leq c_i$ for all $i = 1, \cdots, n$, we assign x with the label $l(x) = n + 1$. Then there exists a unique completely labeled simplex.

Theorem 2.4 *If P has an integral point, then the Labeling Rule results in at least $n + 1$ completely labeled simplices. Moreover, there exists at most one completely labeled simplex of type II.*

Proof: The first part can be derived by induction. The second part follows from the same line of the proof of Theorem 2.2. □

We point out that all theorems above can be rather easily demonstrated by the algorithm to be presented in the next section. Let us give some examples.

Example 1. We are given

$$P = \{ x \in R^2 \,|\, {a_i}^\top x \leq b_i, i = 1,2,3 \}$$

where $a_1 = (3,2)^\top$, $a_2 = (1,-1)^\top$ and $a_3 = (-3,-1)^\top$, $b_1 = 1$, $b_2 = -1$ and $b_3 = 1$. This example is shown in Figure 1 where there are three completely labeled simplices. One of them is of type II. The other two are of type I.

Example 2. We are given

$$P = \{ x \in R^2 \,|\, {a_i}^\top x \leq b_i, i = 1,2,3 \}$$

where $a_1 = (2,-1)^\top$, $a_2 = (3,1)^\top$ and $a_3 = (-3,0)^\top$, $b_1 = 1$, $b_2 = 2$ and $b_3 = -1$. This example is illustrated in Figure 2 where there is a unique completely labeled simplex of type II.

3 The algorithm

In this section we shall discuss how to operate the algorithm in the K_1-triangulation of R^n to find a completely labeled simplex within a finite number of steps. We define a set of $n+1$ vectors of R^n by

$$q(i) = -e(i), i = 1, ..., n$$

and

$$q(n+1) = \sum_{i=1}^{n} e(i),$$

where $e(i)$ denotes the i-th unit vector of R^n, $i = 1,..., n$.

In the rest of the section we assume that the simplex P associated with matrix A is given such that

a. $a_{(n+1)j} \leq 0$ for $j = 1, \cdots, n$;

b. $a_{ii} > 0$ for $i = 1, \cdots, n$; and

c. $a_{ij} \leq 0$ and $\sum_{j=1}^{n} |a_{ij}| < a_{ii}$ for $i \neq j$, $i, j = 1, \cdots, n$.

Such a formulation of the polytope P is referred to as the standard form. Observe that the standard form is rather similar to the Hermite normal form (see e.g., [8, 13]). In the next section we shall show that any n-dimensional simplex P can be restructured into the standard form. Moreover, it is readily to derive the following lemma.

Lemma 3.1 *Let a simplex P be given in the standard form. If P contains two integral points x^1 and x^2, it also contains the integral point*

$$\bar{x} = (\max\{x_1^1, x_1^2\}, \cdots, \max\{x_n^1, x_n^2\})^\top.$$

Now we introduce the K_1-triangulation of R^n (see [4, 14, 15]) which underlies the algorithm. If $x^1 \in Z^n$ and π is a permutation of the set $\{1, 2, ..., n\}$, then denote by $\sigma(x^1, \pi)$ the n-simplex with vertices $x^1, ..., x^{n+1}$ where $x^{i+1} = x^i + e(\pi(i))$ for each $i = 1, ..., n$. Finally, let K_1 be the collection of all such simplices.

Let v be an integral point of R^n. The point v will be the starting point of the algorithm. Define for T being a proper subset of N the regions $A(T)$ by

$$A(T) = \{ x \in R^n \mid x = v + \sum_{j \in T} \lambda_j q(j), \lambda_j \geq 0, j \in T \}.$$

Notice that the dimension of $A(T)$ is t with $t = |T|$. For a proper subset T of N a $(t-1)$-simplex $\sigma(x^1, ..., x^t)$, $1 \leq t \leq n$, is called T-complete if the vertices of σ carry all labels of the set T. Note that every zero-dimensional simplex $\{ y \}$ is $\{ l(y) \}$-complete in case $l(y) \neq 0$.

Now we can formally describe the steps of the algorithm as follows.

Step 0. Set $t = 0$, $x^1 = v$, $T = \emptyset$, $\pi(T) = \emptyset$, $\sigma = \{x^1\}$, $\bar{x} = x^1$, $R_i = 0$, $i = 1, ..., n+1$, and $b = 1$.

Step 1. Calculate $l(x)$ and set $L = l(\bar{x})$. If $L = 0$, an integral point is found and the algorithm terminates. If L is not an element of T, go to Step 3. Otherwise $l(\bar{x}) = l(x^s)$ for exactly one vertex $x^s \neq \bar{x}$ of σ.

Step 2. If $s = t+1$ and $R_{\pi(t)} = 0$, go to Step 4. Otherwise σ and R are adapted according to Table 1 by replacing x^s. Set $b = b+1$. Return to Step 1 with $\bar{x}$ equal to the new vertex of σ.

Step 3. If $t = n$, a completely labeled simplex of type II is found and the algorithm terminates. Otherwise, a $(T \bigcup \{L\})$-complete simplex is found and T becomes $T \bigcup \{L\}$, $\pi(T)$ becomes $(\pi(1), ..., \pi(t), L)$, σ becomes $\sigma(x^1, \pi(T))$ and t becomes $t+1$. Set $b = b+1$. Return to Step 1 with $\bar{x}$ equal to x^{t+1}.

Step 4. Let for some k, $k \leq t$, x^k be the vertex of σ with label $\pi(t)$. Then T becomes $T \backslash \{\pi(t)\}$, $\pi(T)$ becomes $(\pi(1), ..., \pi(t-1))$, σ becomes $\sigma(x^1, \pi(T))$, t becomes $t-1$ and return to Step 2 with $s = k$ and $b = b+1$.

We denote by $E(i)$ the i-th unit vector of R^{n+1}, $i \in N$.

Table 1. Pivot rules if the vertex x^s of $\sigma(x^1, \pi)$ is replaced.

	x^1 becomes	$\pi(T)$ becomes	R becomes
$s = 1$	$x^1 + q(\pi(1))$	$(\pi(2), ..., \pi(t), \pi(1))$	$R + E(\pi(1))$
$1 < s < t+1$	x^1	$(\pi(1), ..., \pi(s), \pi(s-1), ..., \pi(t))$	R
$s = t+1$	$x^1 - q(\pi(t))$	$(\pi(t), \pi(1), ..., \pi(t-1))$	$R - E(\pi(t))$

In order to prove the convergence of the algorithm, we need to borrow some notions from graph theory. First, let us define a graph, denoted by Γ. We say a simplex σ is a node if it satifies one of the following conditions:

(1). $\sigma = \{ v \}$ and $l(v) \neq 0$;

(2). A $(t-1)$-facet, say τ, of σ is a T-complete for a proper subset T of N.

We say two nodes σ_1 and σ_2 in $A(T)$, are adjacent and connected by an edge if one of the following cases occurs:

(1). σ_1 and σ_2 share a common facet τ which is T-complete;

(2). either σ_1 is a T-complete facet of σ_2 or σ_2 is a T-complete facet of σ_1.

Observe that the above relationship is symmetric the edges are not necessarily ordered. Finally, we define the degree of a node σ in the graph by the number of nodes which are adjacent to σ, denoted by $deg(\sigma)$. By adopting the standard argument in van der Laan and Talman [5, 6], we lead to the following observation.

Lemma 3.2 *Let σ be a node in the graph Γ. Then*

(1). $deg(\sigma) = 1$ when $\sigma = \{ v \}$;

(2). $deg(\sigma) = 1$ when σ is a completely labeled simplex; and

(3). $deg(\sigma) = 2$ in all other cases.

Lemma 3.2 implies that the sequence of adjacent simplices of varying dimension starting from the 0-dimensional simplex $\{ v \}$ may lead to a completely labeled simplex, or may terminate with an integral point in P, or may go to infinity. We will prove that the latter case can be excluded.

As norm we use the Euclidean norm in R^n. We now define an open ball of radius γ centered at v by

$$B(\gamma) = \{ x \in R^n \mid ||x - v|| \leq \gamma \}.$$

We have the following lemma.

Lemma 3.3 *For any proper subset T of N, there is no T-complete $(t-1)$-simplex in $A(T)\backslash B(\gamma)$ provided that γ is chosen to be a sufficiently large number.*

Proof: It is a straightforward consequence of the fact that when operating in R^n, the algorithm always moves into the direction in which for some $j \in N$, the function $a_j^\top x - b_j$ is strictly decreasing because of $q(i)^\top a_i < 0$ for all $i \in N$. □.

Now it is easy to obtain the following result by noticing that the number of nodes in the graph Γ is finite and the algorithm will never return to a node previously visited.

Lemma 3.4 *Let an n-simplex P be given in the standard form. Then the algorithm will terminate with an integral point in P or a completely labeled simplex of type II, within a finite number of steps.*

Theorem 3.5 *Let a simplex P be given in the standard form. Then the Labeling Rule precludes the possibility of the coexistence of a completely labeled simplex of type I and a completely labeled simplex of type II.*

Proof: We only need to consider the case in which P contains an integral point, say x^0, i.e., $Ax^0 \leq b$. Let us suppose to the contrary that there is a completely labeled simplex of type II, say $\sigma(x^1, \pi)$. Without loss of generality we may assume that π is equal to $(1, \cdots, n)$. It implies that $l(x^i) = i$ for any $i \in N$. Now it is easy to see that there is a proper subset T of N such that

$$x^1 = x^0 + \sum_{i \in T} r_i q(i),$$

where r_i are positive integers for all $i \in T$. Let us consider the case in which $x^1 = x^0 + kq(s)$ for some positive integer k and some index $s \in N$. The general case can be demonstrated in the same fashion. The following cases need to be addressed:

(1). If $s = 1$, it follows that

$$\begin{aligned} a_1^\top x^1 - b_1 &= a_1^\top(x^0 + kq(1)) - b_1 \\ &= a^\top x^0 - b_1 + ka_1^\top q(1) \\ &\leq -ka_{11} \\ &< 0. \end{aligned}$$

It implies that $l(x^1) \neq 1$. It is a contradiction.

(2). If $1 < s < n+1$, then $x^s = x^0 + \sum_{i=1}^{s-1} q(i) + kq(s)$. We have

$$\begin{aligned} a_s^\top x^s - b_s &= a_s^\top(x^0 + \textstyle\sum_{i=1}^{s-1} q(i) + kq(s)) - b_s \\ &= a_s^\top x^0 - b_s + a_s^\top(kq(s) + \textstyle\sum_{i=1}^{s-1} q(i)) \\ &\leq (-ka_{ss} + \textstyle\sum_{i=1}^{s-1} |a_{si}|) \\ &< 0. \end{aligned}$$

It implies that $l(x^s) \neq s$. It is again a contradiction.

(3). If $s = n+1$, then $x^{n+1} = x^0 + \sum_{i=1}^{n} q(i) + kq(n+1)$. We have

$$\begin{aligned} a_{n+1}^\top x^{n+1} - b_{n+1} &= a_{n+1}^\top(x^0 + \textstyle\sum_{i=1}^{n} q(i) + kq(n+1)) - b_{n+1} \\ &= a_{n+1}^\top x^0 - b_{n+1} + a_{n+1}^\top(kq(n+1) + \textstyle\sum_{i=1}^{n} q(i)) \\ &\leq (k-1)\textstyle\sum_{i=1}^{n} a_{(n+1)i} \\ &\leq 0. \end{aligned}$$

It implies that $l(x^{n+1}) \neq n+1$. It is also a contradiction. □

We now summarize the above discussions.

Theorem 3.6 *The algorithm terminates with either an integral point in P or (exclusively) a completely labeled simplex of type II which shows that there is no integral point in P, within a finite number of iterations.*

Now we sketch a proof for the first part of Theorem 2.4. Let P be given in the standard form. Consider the simplest case in which P contains a single integral point, say w. The general case can be shown in a similar way. Choose w to be the starting point since it is allowed. Take an arbitrary index k from N and set $l(w) = k$ artificially. Then the algorithm will terminate with a completely labeled simplex, say σ_k, within a finite number of steps. If w is a vertex of σ_k, then restore the true label of w, i.e., $l(w) = 0$. Hence σ_k is of type I. Otherwise, it must be of type II. However, this can not happen according to Theorem 3.5. Hence repeat the procedure over all indices of N. We complete the proof.

Let us illustrate the algorithm by some examples.

Example 3. The polytope is given by

$$P = \{\, x \in R^2 \mid a_i^\top x \leq b_i, i = 1, 2, 3 \,\},$$

where $a_1 = (2, -1)^\top$, $a_2 = (-1, 3)^\top$, and $a_3 = (-1, -1)^\top$, $b_1 = 1$, $b_2 = -1$, and $b_3 = 1$. The paths generated by the algorithm lead from $v^1 = (4, -4)^\top$ and $v^2 = (4, 4)^\top$ to the integral point $(0, -1)^\top$ in P, respectively, and are shown in Figure 3.

Example 4. The polytope is given by

$$P = \{\, x \in R^2 \mid a_i^\top x \leq b_i, i = 1, 2, 3 \,\},$$

where $a_1 = (4, -1)^\top$, $a_2 = (0, 1)^\top$, and $a_3 = (-3, 0)^\top$, $b_1 = 1$, $b_2 = 2$, and $b_3 = -1$. The path generated by the algorithm leads from $v = (4, -4)^\top$ to the unique completely labeled simplex of type II and is demonstrated in Figure 4.

4 Reformulation

In order to conform to the standard form, let us come back to the problem. We are given an n-dimensional simplex

$$P = \{\, x \in R^n \mid Ax \leq b \,\},$$

where $a_i^\top = (a_{i1}, ..., a_{in})$ is the i-th row of A for $i \in N$, and $b = (b_1, ..., b_{n+1})^\top$.

A linear transformation U is called a unimodular transformation if U is bijective on Z^n. We recall that U is unimodular if and only if the entries of U are integral and the determinant of U is equal to 1 or -1. Let $\bar{A}$ be the product AU. If U is unimodular, it is readily seen that given an integral point x and $y = Ux$, $Ay \leq b$ if and only if $\bar{A}x \leq b$. Now we present the following transformation theorem.

Theorem 4.1 *For any given n-dimensional simplex*

$$P = \{\, x \in R^n \mid Ax \leq b \,\},$$

there exists a unimodular matrix U such that $\bar{A} = AU$ has the standard form as defined in the previous section.

Proof: It can be easily shown by induction (see e.g., White [16]) that there is a unimodular matrix V such that

$$B = AV = \begin{bmatrix} + & - & \cdots & - & - \\ - & + & \cdots & - & - \\ \vdots & \vdots & & \vdots & \vdots \\ - & - & \cdots & - & + \\ - & - & \cdots & - & - \end{bmatrix}$$

where "+" stands for positive, and "−" for zero or negative, and "−−" for negative in the sequel. Since B is a productive Leontief matrix, there exists a strictly positive integral vector $y \in Z^n$ (i.e., all components of y are positive) satisfying

$$By = z = \begin{bmatrix} + \\ \vdots \\ + \\ -- \end{bmatrix}.$$

Now we can assume that the greatest common divisor of all components of y is 1, denoted by $g.c.d(y_1, \cdots, y_n) = 1$. In case $y_1 = \cdots = y_n = 1$, let $U = V$. We are done. Otherwise, it is clear that there exists a unimodular matrix W such that $y = Wq(n+1)$. Notice that W does not change the sign of the components of $q(n+1)$. Now it follows from the basic geometry theory that there is a unimodular matrix Z such that

$$\bar{A} = BWZ$$

keeps the same sign as B and

$$BWZW^{-1}y = \bar{z} = \begin{bmatrix} + \\ \vdots \\ + \\ -- \end{bmatrix}.$$

Let $U = VWZ$. We complete the proof. $\square$

We remark that the origin of R^n is still contained in the interior of the convex hull of the vectors $\bar{a}_1, \ldots, \bar{a}_{n+1}$. For $n = 2$, we can construct such unimodular matrices by adapting Scarf's method in [10]. Now the algorithm can be applied.

Let us give some examples.

Example 5. We are given

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Let $b_1 = 0$, $b_2 = 1$, and $b_3 = 0$. That is,

$$P = \{\, x \in R^2 \mid {a_i}^\top x \leq b_i, i = 1, 2, 3 \,\}.$$

Example 6. We are given

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ -3 & -1 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \\ -1 & -1 \end{bmatrix}.$$

Let $b_1 = 1$, $b_2 = -1$, and $b_3 = 1$. Observe that this example is taken from Example 1 and Example 3, respectively. See Figure 3 where there are three completely labeled simplices of type I after having been transformed. In this case there is no completely labeled simplex of type II. The reader should compare Figure 1 with Figure 3.

Example 7. We are given

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ -3 & 0 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 4 & -1 \\ 0 & 1 \\ -3 & 0 \end{bmatrix}.$$

Let $b_1 = 1$, $b_2 = 2$ and $b_3 = -1$. Then A and $\bar{A}$ correspond to Example 2 and Example 4, respectively. See Figure 2 and Figure 4.

Example 8. We are given

$$P = \{\, x \in R^2 \,|\, {a_i}^\top x \le b_i, i = 1, 2, 3 \,\}$$

where $a_1 = (3, -1)^\top$, $a_2 = (-3, 2)^\top$ and $a_3 = (-1, -1)^\top$, $b_1 = 2$, $b_2 = -1$ and $b_3 = 0$. See Figure 5 where there are three completely labeled simplices. One of them is of type II. The other two are of type I. We use the following unimodular transformation

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ -2 & -1 \end{bmatrix}.$$

Now it is easy to check that the resulting polytope generates no completely labeled simplex of type II. In fact the Labeling Rule results in three completely labeled simplices of type I in this case, see Figure 6. Compare Figure 5 with Figure 6.

Example 9. We are given

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

such that

$$\bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \\ -3 & -2 & -2 \end{bmatrix}.$$

Let $b_1 = 1$, $b_2 = b_3 = 2$, and $b_4 = -1$. Then we get the corresponding polytopes.

One may wonder how to get a unimodular matrix. Clearly, the identity I_n is unimodular. In order to obtain a non-trivial unimodular matrix, however, we need to study some other examples and their effect when they postmultiply a matrix A and premultiply a vector x.

(*i*) Interchange: U is equal to I_n except that the k-th column of U is $e(l)$ and the l-th column of U is $e(k)$. This transformation switches columns k and l of A and switches the k-th and l-th components of x.

(*ii*) Reversal of sign: U is equal to I_n except that the k-th column of U is equal to $-e(k)$. U changes the sign of the entries of the k-th column of A and of the k-th component of x.

(*iii*) Addition: U is equal to I_n except that the (k,l)-th entry of U, $k \neq l$, is equal to one. This transformation replaces the l-th column of A by the sum of columns k and l and replaces the k-th component of x by the sum of components k and l.

The following result can be found in Newman [9].

Theorem 4.2 *Every unimodular matrix can be expressed as a finite product of unimodular matrices of type (i), (ii) and (iii).*

Now let us give a proof of Theorem 2.2. We are given a polytope which has the standard form. Since P contains no integral point, the algorithm terminates with a completely labeled simplex of type II, say, $\sigma_1(x^1, \pi)$, within a finite number of steps. Suppose to the contrary that there is another completely labeled simplex of tpye II, say, $\sigma_2(y^1, \rho)$. Without loss of generality we may assume that π is equal to $(1, ..., n)$ and $x^i = y^i$ for all $i \in N$ except for some index k, $1 \leq k \leq n+1$. It implies that $l(x^i) = l(y^i) = i$ for all $i \in N$. We have to consider the following cases:

(1). If $k = 1$, then $y^1 = x^{n+1} + q(1)$. Since $l(x^{n+1}) = n+1$, we have

$$a_{n+1}^\top x^{n+1} - b_{n+1} > a_i^\top x^{n+1} - b_i, i = 1, \cdots, n.$$

Further, we have

$$\begin{aligned} a_{n+1}^\top y^1 - b_{n+1} &= a_{n+1}^\top (x^{n+1} + q(1)) - b_{n+1} \\ &\geq a_{n+1}^\top x^{n+1} - b_{n+1} \\ &> a_1^\top x^{n+1} - b_1 \\ &> a_1^\top (x^{n+1} + q(1)) - b_1 \\ &= a_1^\top y^1 - b_1. \end{aligned}$$

It means that $l(y^1) \neq 1$. It is a contradiction.

(2). If $1 < k < n+1$, then $y^k = x^{k-1} + q(k)$. Since $l(x^{k-1}) = k-1$, it implies that

$$a_{k-1}^\top x^{k-1} - b_{k-1} \geq a_k^\top x^{k-1} - b_k.$$

Hence we have

$$\begin{aligned} a_{k-1}^\top y^k - b_{k-1} &= a_{k-1}^\top (x^{k-1} + q(k)) - b_{k-1} \\ &= a_{k-1}^\top x^{k-1} - b_{k-1} + a_{k-1}^\top q(k) \\ &\geq a_k^\top x^{k-1} - b_k + a_k^\top q(k) \\ &\geq a_k^\top (x^{k-1} + q(k)) - b_k \\ &= a_k^\top y^k - b_k. \end{aligned}$$

It means that $l(y^k) \neq k$. It is again a contradiction.

(3). If $k = n + 1$, then $y^{n+1} = x^1 - q(n)$. Notice that

$$a_n^\top x^1 - b_n \geq a_{n+1}^\top x^1 - b_{n+1}.$$

We have

$$\begin{aligned} a_n^\top y^{n+1} - b_n &= a_n^\top (x^1 - q(n)) - b_n \\ &\geq a_{n+1}^\top x^1 - b_{n+1} - a_n^\top q(n) \\ &> a_{n+1}^\top x^1 - b_{n+1} - a_{n+1}^\top q(n) \\ &= a_{n+1}^\top (x^1 - q(n)) - b_{n+1} \\ &= a_{n+1}^\top y^{n+1} - b_{n+1}. \end{aligned}$$

It means that $l(y^{n+1}) \neq n + 1$. It is also a contradiction. $\square$.

5 Extension to general n-dimensional polytopes

Let I_m denote the set of integers $\{ 1, \cdots, m \}$. The problem is to test the integral property of a general n-dimensional polytope P given by

$$P = \{ x \in R^n \mid Ax \leq b \},$$

where $a_i{}^\top = (a_{i1}, ..., a_{in})$ is the i-th row of the m by n matrix A for $i = 1$, ..., m, and $b = (b_1, \cdots, b_m)^\top$ is a vector of R^m. It is clear that $m \geq n + 1$. As usual we may assume that a_1,..., a_m are integral vectors of R^n, and $b = (b_1, \cdots, b_m)^\top$ is an integral vector of R^m. Finally we assume that none of the constraints $a_i^\top x \leq b_i$, $i \in I_m$, is redundant, and that there is a subset J, with cardinality $n + 1$, of I_m such that

$$\{ x \in R^n \mid a_i^\top x \leq b_i, i \in J \}$$

is an n-dimensional simplex. In the sequel we take $J = N$ for simplicity of notation.

Compared with the labeling rule in Section 2, we have the following generalized labeling rule.

Generalized Labeling Rule: If there is an index $i \in I_m$ for which $a_i^\top x - b_i > 0$, we assign $x \in Z^n$ with the label

$$l(x) = \min\{\, j \in N \,|\, a_j^\top x - b_j = \max_{h \in N}\{\, a_h^\top x - b_h \}\, \}.$$

If $a_h^\top x \leq b_h$ for all $h \in I_m$, then $l(x) = 0$.

Similarly, we have the following results.

Theorem 5.1 *If P does not contain any integral point, then the Generalized Labeling Rule results in a unique completely labeled simplex. Moreover, the unique completely labeled simplex must be of type II.*

Theorem 5.2 *If P contains an integral point, then the Generalized Labeling Rule results in at least $n+1$ completely labeled simplices. Moreover, there exists at most one completely labeled simplex of type II.*

Correspondingly, we can reformulate any n-dimensional polytope P into the standard form: namely, the first $n+1$ rows of the matrix A satisfy the conditions (a), (b) and (c) as defined in Section 3. Observe that the standard form does not impose any condition on the constraint vectors a_i for $i \in I_m \backslash N$. This indicates that if we transform the problems into the standard form, we only need to focus on the first $n+1$ constraint vectors and then postmultiply the rest constraint vectors by the resulting unimodular matrix U.

Let us give some examples.

Example 10. We are given

$$P = \{\, x \in R^2 \,|\, a_i^\top x \leq b_i, i = 1, \cdots, 5\, \}$$

where $a_1 = (2,-1)^\top$, $a_2 = (-1,3)^\top$, $a_3 = (-1,-2)^\top$, $a_4 = (2,1)^\top$, and $a_5 = (-2,0)^\top$, $b_1 = 1$, $b_2 = 3$, $b_3 = 2$, $b_4 = 2$, and $b_5 = 3$. This example is shown in

Figure 7 where there are three completely labeled simplices of type I.

Example 11. We are given

$$P = \{ x \in R^2 \mid a_i^\top x \leq b_i, i = 1, \cdots, 5 \}$$

where $a_1 = (3,-2)^\top$, $a_2 = (-1,4)^\top$, $a_3 = (-5,-8)^\top$, $a_4 = (3,2)^\top$, and $a_5 = (0,-4)^\top$, $b_1 = 2$, $b_2 = 3$, $b_3 = -4$, $b_4 = 4$, and $b_5 = -1$. This example is illustrated in Figure 8 where there is a unique completely labeled simplex of type II.

Example 12. We are given

$$P = \{ x \in R^2 \mid a_i^\top x \leq b_i, i = 1, \cdots, 5 \}$$

where $a_1 = (3,-2)^\top$, $a_2 = (-1,2)^\top$, $a_3 = (-1,-2)^\top$, $a_4 = (0,5)^\top$, and $a_5 = (0,-5)^\top$, $b_1 = 6$, $b_2 = 2$, $b_3 = 0$, $b_4 = 4$, and $b_5 = -1$. This example is depicted in Figure 9 where there is a unique completely labeled simplex of type II.

Example 13. We are given

$$P = \{ x \in R^2 \mid {a_i}^\top x \leq b_i, i = 1, 2, 3, 4 \}$$

where $a_1 = (2,-1)^\top$, $a_2 = (-1,2)^\top$ $a_3 = (0,-2)^\top$, and $a_4 = (-10,0)^\top$, $b_1 = 4$, $b_2 = -2$, $b_3 = 3$, and $b_4 = 11$. This example is shown in Figure 10 where there are three completely labeled simplices. One of them is of type II. The other two are of type I. The paths generated by the algorithm lead from $v^1 = (-4,-2)^\top$ and $v^2 = (4,-4)^\top$ to the unique completely labeled simplex of type II. While the path generated by the algorithm leads from $v^3 = (4,2)^\top$ to a feasible integral point $(2,0)^\top$. In the following, we shall call the algorithm in Section 3 the Basic algorithm.

From the previous example, the reader may notice that the Basic algorithm does not provide a definite answer in case it terminates with a completely labeled simplex of type II. In order to get a definite answer, let us define for $k \in N$ a subset C_k of Z^n by

$$C_k = \{ x \in Z^n \mid a_j^\top x > b_j \text{ for } j \in N_{-k} \}.$$

Now we establish our Global procedure.

Step(1). Set $k = 1$.

Step(2). Choose a starting point $v^k \in C_k$ and implement the Basic algorithm. If an integral point in P is found, then stop. Otherwise, k becomes $k + 1$.

Step(3). If $k = n + 2$, then stop. Otherwise, go to Step (2).

The procedure is shown in Figure 10 by Example 13 where $v^1 \in C_1$, $v^2 \in C_2$, and $v^3 \in C_3$. The procedure leads to the integral point $(2, 0)^\top$ which gives a definite answer to the problem. Now we turn to discuss how to obtain a starting point $v^k \in C_k$. For each $k \in N$, we define

$$\bar{q}(k) = -q(k).$$

For $x \in Z^n$, we assign x with the label

$$\bar{l}(x) = \min\{\, j \in N_{-k} \mid a_j^\top x - b_j = \min\{\, a_h^\top x - b_h, h \in N_{-k} \mid a_h^\top x - b_h \le 0 \,\}\}.$$

If $a_h^\top x - b_h > 0$ for all $h \in N_{-k}$, then $\bar{l}(x) = 0$.

Now we can apply the Basic algorithm by using $\bar{l}(.)$ and $\bar{q}(.)$ instead of $l(.)$ and $q(.)$. It is easy to verify that the Basic algorithm will find an integral point $v^k \in C_k$ within a finite number of steps. We illustrate the algorithm in Figure 11 by Example 13. Take for instance $k = 3$ and $v = (-2, -3)^\top$. The algorithm finds an integral point in C_3, namely, $(3, 1)^\top$.

We conclude with the following observation.

Theorem 5.3 *Given a polytope P in the standard form, the Global procedure terminates either with an integral point in P or a completely labeled simplex of type II which proves there is no integral point in P, within a finite number of steps.*

6 Extension to general lower dimensional polytopes

In the previous section we assume that the polytope P in R^n is n-dimensional and that none of the constraints $a_i^\top x \leq b_i$ is redundant. If the set P is lower-dimensional set in R^n we may assume under the same conditions that P can be expressed as

$$P = \{\, x \in R^n \,|\, a_i^\top x \leq b_i, i = 1, \cdots, m, \text{ and } c_i^\top x = d_i, i = 1, \cdots, m^1 \,\},$$

where $dimP = n - m^1$, for some m^1, $0 \leq m^1 \leq n$. As usual we assume that a_1,..., a_m, c_1,..., c_{m^1} are integral vectors of R^n, and b_1, $\cdots$, b_m, d_1,...,d_{m^1} are integers. Let A denote an $m \times n$ matrix whose rows are $a_1^\top$, $\cdots$, $a_m^\top$, and let $b = (b_1, \cdots, b_m)^\top$. Moreover, let C be an $m^1 \times n$ matrix whose rows are $c_1^\top$, $\cdots$, $c_{m^1}^\top$, and let $d = (d_1, \cdots, d_{m^1})^\top$. We define a set Q by

$$Q = \{\, x \in Z^n \,|\, Cx = d \,\}.$$

It is clear that if Q is empty, then P has no integral point.

In such cases we can transform the polytopes into full low-dimensional ones by reducing the number of variables in polynomial time. To do so, we first introduce one definition.

Definition 6.1 *An $m^1 \times m^1$ nonsingular integer matrix H is called in Hermite normal form if it satisfies the following conditions:*

(1). H is lower triangular and $h_{ij} = 0$ for $i < j$;

(2). $h_{ii} > 0$ for $i = 1, \cdots, m^1$; and

(3). $h_{ij} \leq 0$ and $|h_{ij}| < h_{ii}$ for $i > j$.

The following two results can be found in e.g., [8, 13].

Theorem 6.2 *Given the matrix C, there exists an $n \times n$ unimodular matrix U such that*

(i). $CU = (H, 0)$ and H is in Hermite normal form;

(ii). $H^{-1}C$ is an integer matrix.

A polynomial-time algorithm for finding U and H can be also found in [8, 13]. Now let H and $U = (U_1, U_2)$ be as in Theorem 6.2, with an $n \times m^1$ matrix and U_2 an $n \times (n - m^1)$ matrix.

Theorem 6.3

(i). Q is nonempty if and only if $H^{-1}.d \in Z^n$.

(ii). If Q is nonempty, every solution of Q can be written as

$$x = U_1.H^{-1}.d + U_2.z, \;\; z \in Z^{n-m^1}.$$

In case $Q = \emptyset$, P contains no integral point. While Q is nonempty, we have

$$\begin{aligned}
P_0 &= \{ y \in R^n \;|\; AUy \le b, \text{ and } CUy = d \} \\
&= \{ y \in R^n \;|\; AUy \le b, \text{ and } [H, 0]y = d \} \\
&= \{ y \in R^n \;|\; y = ((H^{-1}d)^\top, z^\top)^\top, \text{ and } AU((H^{-1}d)^\top, z^\top)^\top \le b, z \in R^{n-m^1} \} \\
&= \{ y \in R^n \;|\; y = ((H^{-1}d)^\top, z^\top)^\top, \text{ and } \bar{A}z \le \bar{b}, \, z \in R^{n-m^1} \}.
\end{aligned}$$

Let

$$\bar{P} = \{ z \in R^{n-m^1} \,|\, \bar{A}z \le \bar{b} \}.$$

Doing so leads to an $(n - m^1)$-dimensional polytope $\bar{P}$ in R^{n-m^1}. Now the rest discussions are the same as in the previous section. Let us demonstrate by one example as follows: We are given a polytope

$$P = \{ x \in R^3 \,|\, a_i^\top x \le b_i, i = 1, 2, 3, \text{ and } c_1^\top x = d_1 \},$$

where $a_1 = (-1, 0, 0)^\top$, $a_2 = (0, -1, 0)^\top$, $a_3 = (0, 0, -1)^\top$, $c_1 = (4, 12, 2)^\top$, $b_1 = 0$, $b_2 = 0$, $b_3 = 1$, and $d_1 = 2$. Then we have

$$C = [4\ 12\ 2],$$

$$U = \begin{bmatrix} 1 & 3 & -7 \\ 0 & -1 & 2 \\ -1 & 0 & 2 \end{bmatrix},$$

$$H = [2],$$

and

$$CU = [2\ 0\ 0].$$

Moreover, we decompose U as $U = (U_1, U_2)$

$$U_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

and

$$U_2 = \begin{bmatrix} 3 & -7 \\ -1 & 2 \\ 0 & 2 \end{bmatrix}.$$

Notice that $y_1 = 1$. We get a 2-dimensional polytope

$$\bar{P} = \{z \in R^2 \mid \bar{A}z \leq \bar{b}\}$$

where

$$\bar{A} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \\ 0 & -2 \end{bmatrix},$$

and $\bar{b} = (1, 0, 0)^\top$.

7 Concluding remarks

Some preliminary numerical results indicate that the algorithm works remarkably well. A large number of large-scale instances (more than $100,000$ variables) is also being carried out. The instances can be easily created according to the standard form. The reader should have no difficulty in making such a program by himself. We shall report numerical results and the complexity analysis of the algorithm in the next paper.

Here we do not discuss the index theory. In fact the reader can build up an index theory for the algorithm by himself. The interested reader is encouraged to read van der Laan [4] and Scarf [11] for insightful discussions. Finally, we point out that the future research shall be focused on how to efficiently transform the problems into the standard form.

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References

[1] C. Dang and H. van Maaren, "A simplicial approach to integer programming", Report 93-06, Department of Mathematics, Delft University of Technology, 1993, Holland, to appear in *Mathematics of Operations Research.*

[2] B.C. Eaves, "Homotopies for computation of fixed points", *Mathematical Programming* 3 (1972) 1-22.

[3] H.W. Kuhn, "Simplicial approximation of fixed points", *Proceedings of National Academy of Science* 61 (1968) 1238-1242.

[4] G. van der Laan, Simplicial Fixed Point Algorithms, CWI, Amsterdam, 1981.

[5] G. van der Laan and A.J.J. Talman, "A restart algorithm for computing fixed points without an extra dimension", *Mathematical Programming* 17 (1979) 74-84.

[6] G. van der Laan and A.J.J. Talman, "A class of simplicial restart fixed point algorithms without an extra dimension", *Mathematical Programming* 20 (1981) 33-48.

[7] G. van der Laan, A.J.J. Talman and L. Van der Heyden, "Simplicial variable dimension algorithms for solving the nonlinear complementarity problems on a product of unit simplices using a general labeling", *Mathematics of Operations Research* 12 (1987) 377-397.

[8] G.L. Nemhauser and L.A. Wolsey, Integer and Combinatorial Optimization, John Wiley & Sons, New York, 1988.

[9] M. Newman, Integral Matrices, Academic Press, New York, 1972.

[10] H. Scarf, "Production sets with indivisibilities-part II: the case of two activities", *Econometrica* 49 (1981) 395-423.

[11] H. Scarf, "Production sets with indivisibilities-part I: generalities", *Econometrica* 49 (1981) 1-32.

[12] H. Scarf, The Computation of Economic Equilibria, Yale University Press, New Haven, 1973.

[13] A. Schrijver, Theory of Linear and Integer Programming, John Wiley & Sons, New York, 1986.

[14] A.J.J. Talman, Variable Dimension Fixed Point Algorithms and Triangulations, CWI, Amsterdam, 1981.

[15] M.J. Todd, The Computation of Fixed Points and Applications, Lecture Notes in Economics and Mathematical Systems 124, Springer-Verlag, Berlin, 1976.

[16] P.M. White, Discrete Activity Analysis, Ph.D. Thesis, Yale University, New Haven, 1983.

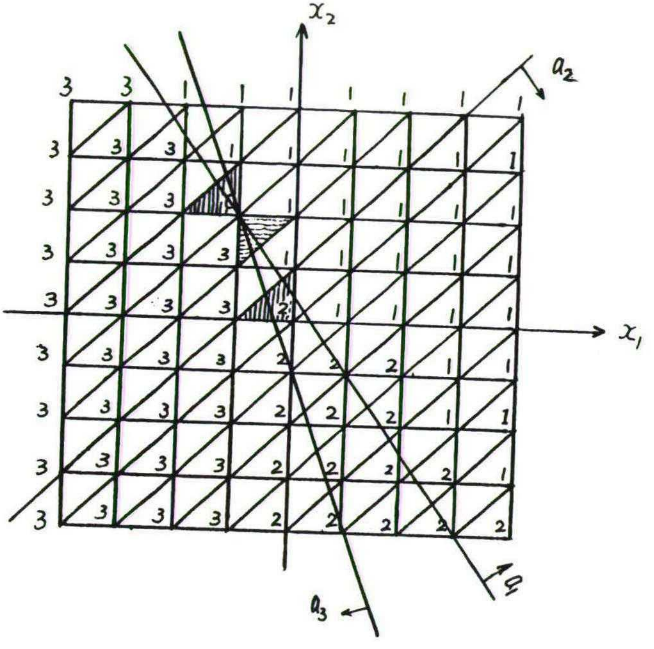

Figure 1. Three completely labeled simplices in Example 1.

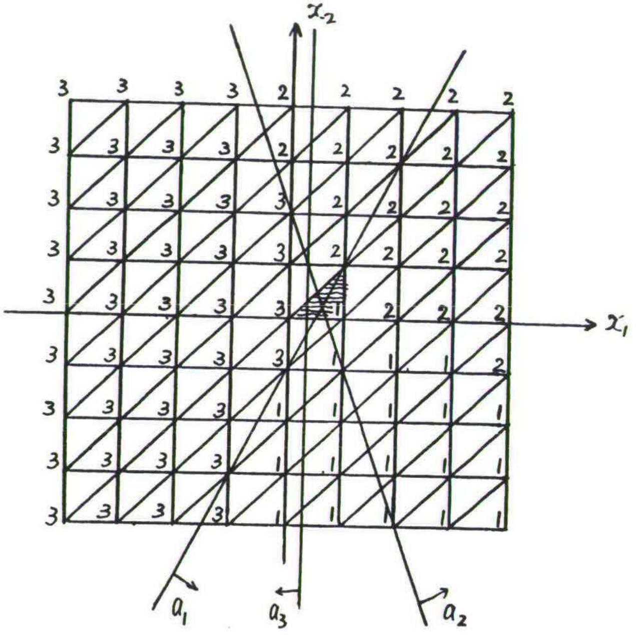

Figure 2. A unique completely labeled simplex in Example 2.

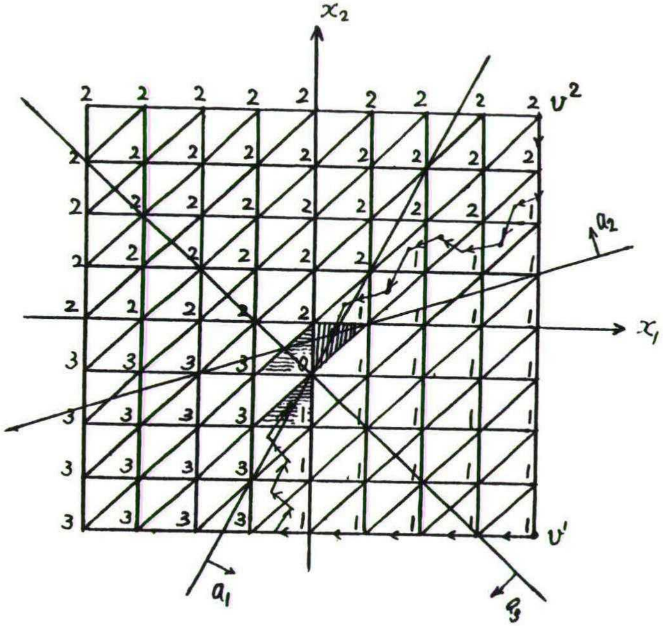

Figure 3. The paths generated by the algorithm lead to a lattice point in P for Example 3.

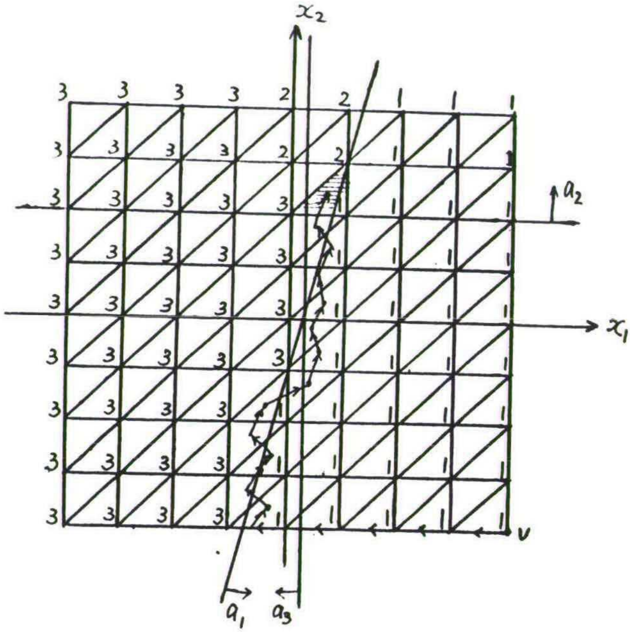

Figure 4. The algorithm converges to the unique completely labeled simplex in Example 4.

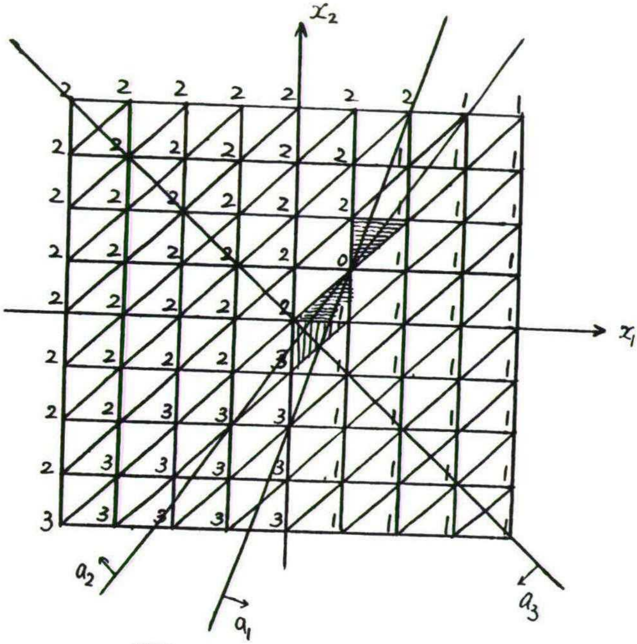

Figure 5. Three completely labeled simplices, two of type I, in Example 8 before transformed.

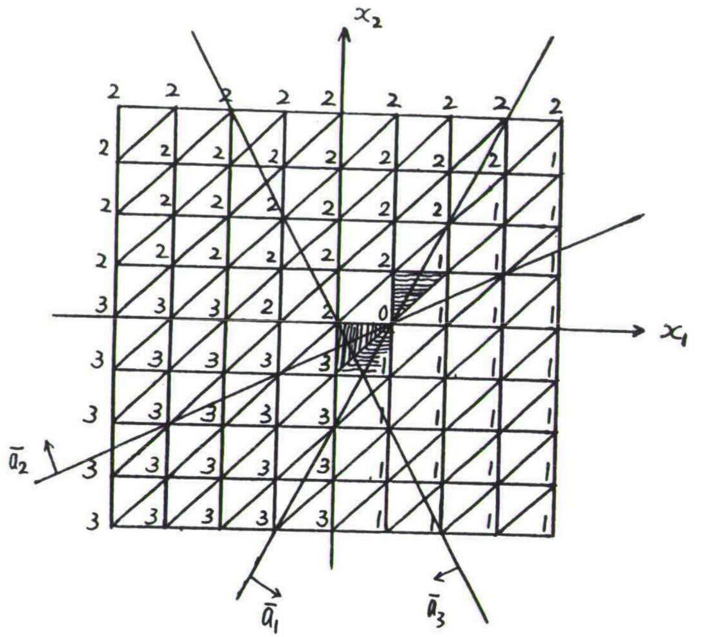

Figure 6. Three completely labeled simplices of type I, in Example 8 after transformed.

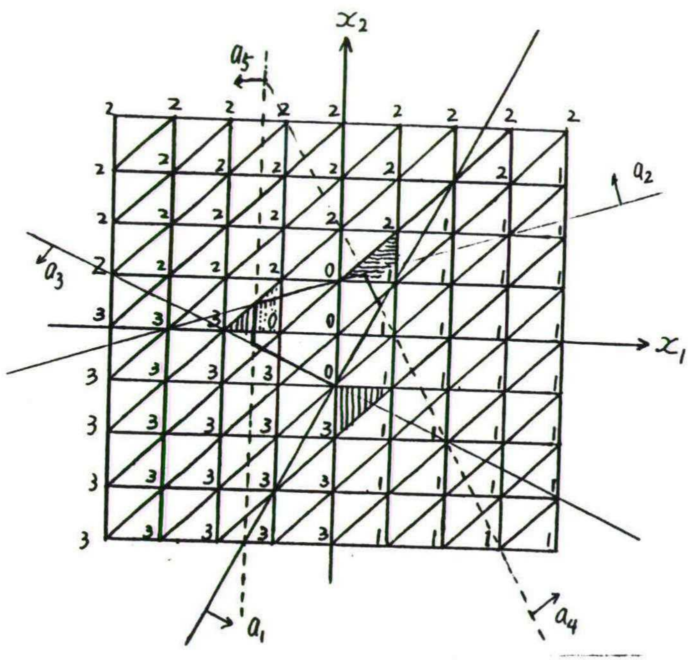

Figure 7. Three completely labeled simplices of type I in Example 10.

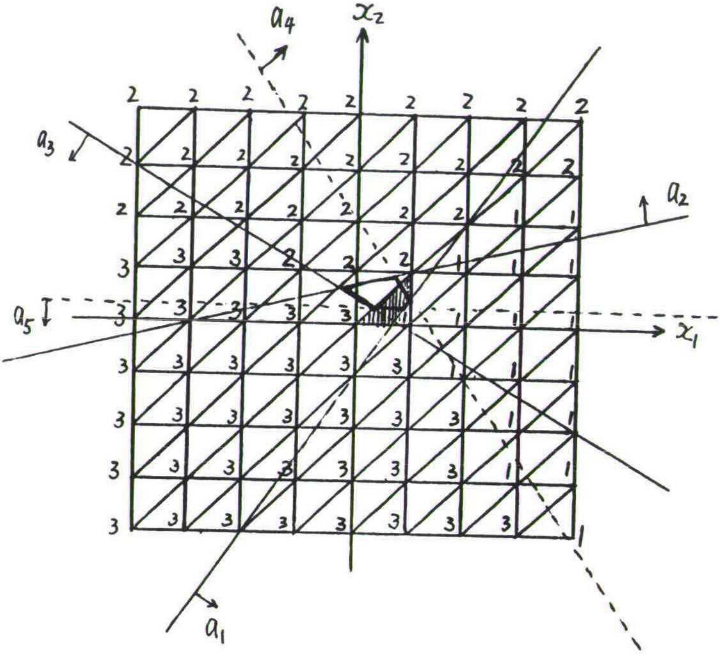

Figure 8. A unique completely labeled simplex in Example 11.

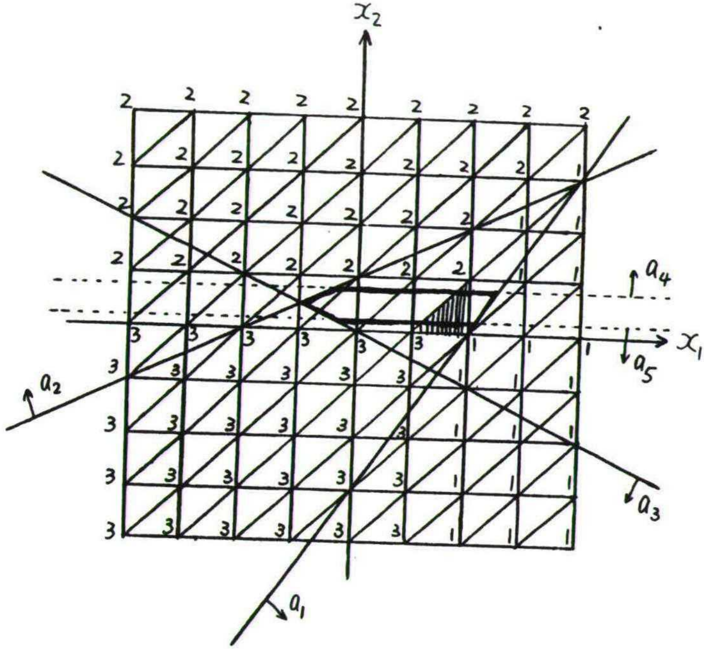

Figure 9. A unique completely labeled simplex in Example 12.

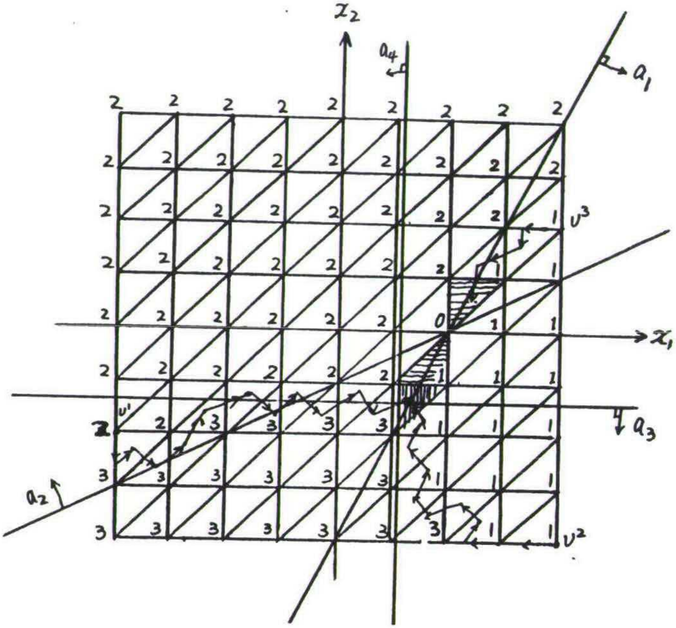

Figure 10. Three completely labeled simplices in Example 13.

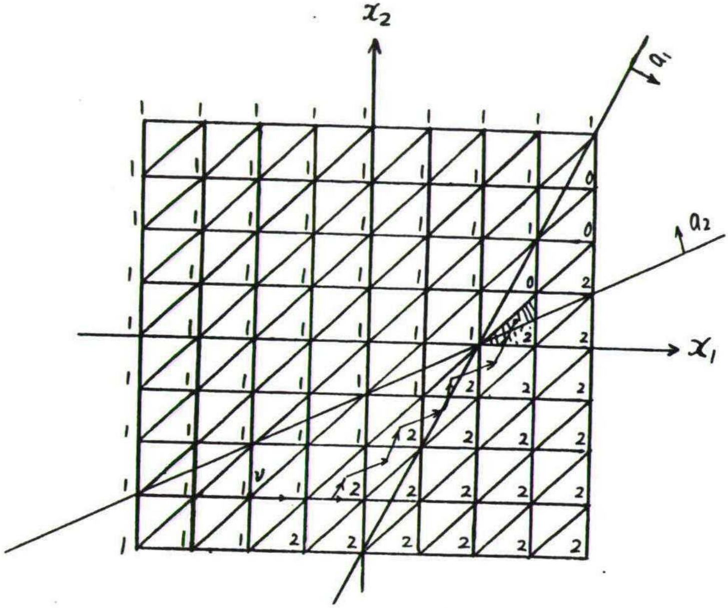

Figure 11. The path leads to an integral point in C_3 in Example 13.

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